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## NOTES ON NORMAL DISTRIBUTION

**Abstract.** In this paper several characteristics of normal distribution are presented. A new bimodal Weber distribution model is also postulated.

**Key words:** normal distribution, Cramer's theorem, Feller's and Novak's characterizations, Webber's distribution.

**1.** Normal distribution is one of the most important probability distribution formulas used in the theory and practice of probability science and statistics. Normal distribution was originally introduced by de Moivre (H. Cramer (1958)) in 1733, in his examination of limes forms of binomial distribution. This initial postulate went largely unnoticed, leading to the re-discovery of normal distribution in the works of Gauss in 1809 and Laplace in 1812 (H. Cramer (1958)). The authors arrived at normal distribution principles in the course of their analyses of experiment error theory.

**Definition 1.** *Random variable  $X$  is considered as falling into normal distribution with parameters  $m$  and  $s$  (in principle,  $X \sim N(m, s)$ ), where  $s > 0$ ,  $m \in R$ , if its density function takes the form of*

$$f_{m,s}(X) = \frac{1}{s\sqrt{2\pi}} \exp\left(-\frac{(x-m)^2}{2s^2}\right), \quad x \in R.$$

As seen in the above formula, the resulting curve is symmetrical and unimodal, reaching maximum at point  $x = m$ , which at the same time is the mean ( $E(X) = m$ ), median and modal value of the distribution. Variance of random variable  $X$  is expressed by the second parameter:

$$\text{Var}(X) = s^2.$$

In this case, the moments of odd order in relation to the mean equal zero:

$$E(X - m) = E(X - m)^3 = E(X - m)^5 = \dots = 0,$$

while the moments of even order in relation to the mean equal:

$$E(X - m)^{2n} = 1 \cdot 3 \cdot \dots \cdot (2n - 1)s^{2n}, n \in N.$$

The figure below shows the normal distribution density function for  $m = 0, s = 1$ .

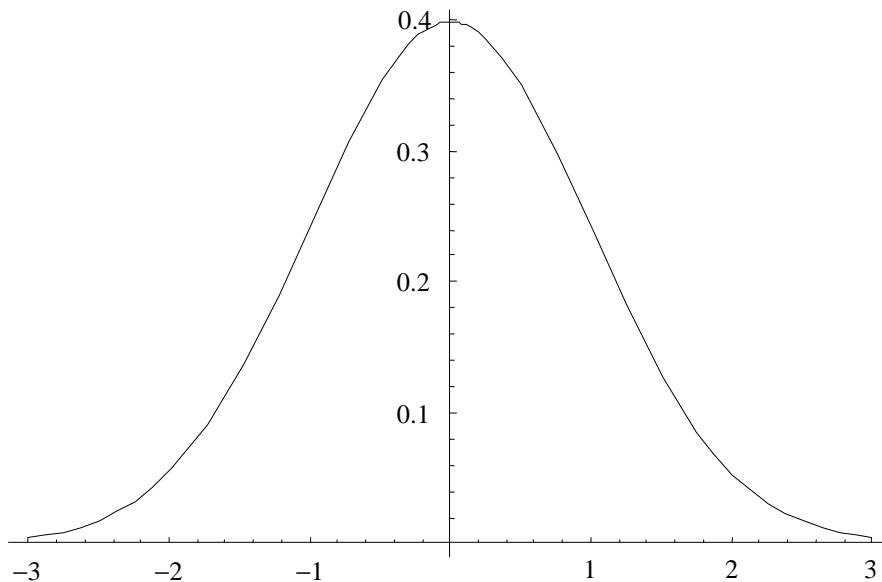


Fig. 1

Source: own research.

**2.** This section addresses selected properties that characterize normal distribution. Variants of the central border theorem as well as the infinite divisibility property are omitted, with discussion centered on some of the less-known characteristics.

- If  $U$  and  $V$  are independent random variables defined on the same probability space, monotonously distributed on  $(0, 1)$ , then

$$X = \sqrt{-2 \log U} \cos(2\pi V)$$

and

$$Y = \sqrt{-2\log U} \sin(2\pi V)$$

are independent and distributed along  $N(0, 1)$  (J. Jakubowski, R. Sztencel (2000)). This property is frequently used in normal distribution random number generators (random numbers in monotonous distribution can be generated fairly easily).

- Cramér's theorem. If normally distributed random variable  $X$  is a sum of two independent random variables  $Y$  and  $Z$ , then those both variables are normally distributed as well (H. Cramer (1958)).

- Let random variables  $U, V$  be independent, and

$$X = aU + bV,$$

$$Y = cU + dV.$$

If  $X$  and  $Y$  are independent, then all four variables are normal, unless  $b = c = 0$  or  $a = d = 0$  (W. Feller (1978)).

It must be noted, that the above property allows to define Gaussian random variables in infinite-dimension Banach spaces or groups (in the latter case, it is enough to define the sum).

- Let  $R(X, Y)$  be defined as:

$$R(X, Y) = \sup r\{f(X), g(Y)\},$$

where  $r$  is a correlation coefficient of respective random variables, while supremum applies to all functions  $f$  and  $g$ , for which

$$0 < \text{Var}\{f(X)\} < \infty,$$

$$0 < \text{Var}\{g(Y)\} < \infty.$$

If random vector  $(X, Y)$  is normal, then

$$R(X, Y) = |r(X, Y)|.$$

The proof of this theorem can be found in (H.O. Lancaster (1957); Y. Yu (2008)).

- Assume that random variables  $X$  and  $Y$  are independent and identically distributed. Then

$$\frac{2XY}{\sqrt{X^2 + Y^2}} \sim N(0, s)$$

only if  $X \sim N(0, s)$ .

Analogical characteristics applies to symmetrical Bernoulli distribution. While keeping the above assumptions,

$$\frac{2XY}{\sqrt{X^2 + Y^2}}$$

is a random variable which is symmetrically, Bernoulli distributed only if variable  $X$  is distributed in the same way. It is worth noting that Poisson distribution and standard Bernoulli distribution do not share the above property. For proof on that see S.Y. Novak (2007).

**3.** Normal distribution is closely related to some other distribution patterns widely used in statistics. These include:

- Chi-square distribution,
- Student's t-distribution,
- Fisher's z-distribution.

This section includes postulated bimodal distribution of smooth density plot, other than a mix of distributions defined on a straight line.

**Definition 2.** Random variable  $X$  is said to be Weber-distributed with parameters  $\alpha, \beta, \gamma$  (i.e.  $X \sim W(\alpha, \beta, \gamma)$ ), where  $\alpha, \beta > 0, \gamma \in \mathbb{R}$ , if the density function is in the form of

$$g_{\alpha, \beta, \gamma}(x) = \frac{1}{z(\alpha, \beta)} \exp\left(\alpha(x - \gamma)^2 - \beta(x - \gamma)^4\right), \quad x \in \mathbb{R}.$$

It is interesting to note here the normalization constant  $z(\alpha, \beta)$ . As it turns out, the above integral may be expressed using Weber special functions (H. Bateman, A. Erdelyi (1953)), that are arrayed. Thence:

$$z(\alpha, \beta) = \int_{\mathbb{R}} e^{\alpha x^2 - \beta x^4} dx = \exp\left(\frac{\alpha^2}{8\beta}\right) \frac{\sqrt{\pi}}{\sqrt[4]{2\beta}} D_{-\frac{1}{2}}\left(-\frac{\alpha}{\sqrt{2\beta}}\right),$$

where  $D$  is a Weber (N.A. Weber (1946)) function that satisfies the differential equation of:

$$\frac{d^2 D_p(x)}{dx^2} + \left(p + \frac{1}{2} - \frac{x^2}{4}\right) D_p(x) = 0,$$

for  $p \neq 0, 1, 2, \dots$

As seen from the above definition, the  $g$  function is symmetrical, with two mods (maxima) in points

$$Mo_1 = -\sqrt{\frac{\alpha}{2\beta}} + \gamma,$$

$$Mo_2 = \sqrt{\frac{\alpha}{2\beta}} + \gamma,$$

while the expected value of random variable  $X$  is:

$$E(X) = \gamma.$$

The resulting moments of odd order in relation to mean equal zero:

$$E(X - \gamma) = E(X - \gamma)^3 = E(X - \gamma)^5 = \dots = 0,$$

while the moments of even order in relation to mean equal:

$$E(X - \gamma)^{2n} = \frac{1}{2z(\alpha, \beta)} \beta^{\frac{1}{4}(3-2n)} \left( \begin{array}{l} \sqrt{\beta} \Gamma\left(\frac{1}{4} + \frac{n}{2}\right) H\left(\frac{1}{4} + \frac{n}{2}, \frac{1}{2}, \frac{\alpha^2}{4\beta}\right) + \\ + \alpha \Gamma\left(\frac{3}{4} + \frac{n}{2}\right) H\left(\frac{3}{4} + \frac{n}{2}, \frac{3}{2}, \frac{\alpha^2}{4\beta}\right) \end{array} \right),$$

where  $n = 1, 2, \dots$  and  $H$  is a confluent hypergeometric function (H. Bateman, A. Erdelyi (1953)).

The function is expressed as a sum of the series:

$$H(a, b, x) = \sum_{k=0}^{\infty} \frac{(a)_k x^k}{(b)_k k!},$$

$a, b, x \in \mathfrak{R}$ ,

$$(a)_k = (a+k-1)(a+k-2)\dots(a+2)(a+1).$$

It can also be expressed in the integral form of:

$$H(a, b, x) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{b-a-1} dt.$$

The confluent hypergeometric function is a solution of Kummer differential equation:

$$xy'' + (b - x)y' - ay = 0$$

with boundary conditions

$$y(0) = 1$$

and

$$y'(0) = \frac{a}{b}.$$

It can be considered a generalization of some other special functions, such as: Weber function, Bessel's function, Laguerre's and Hermite's polynomials, etc. Its values are arrayed.

Below, a distribution density function is shown,  $W(2, 1, 1)$ .

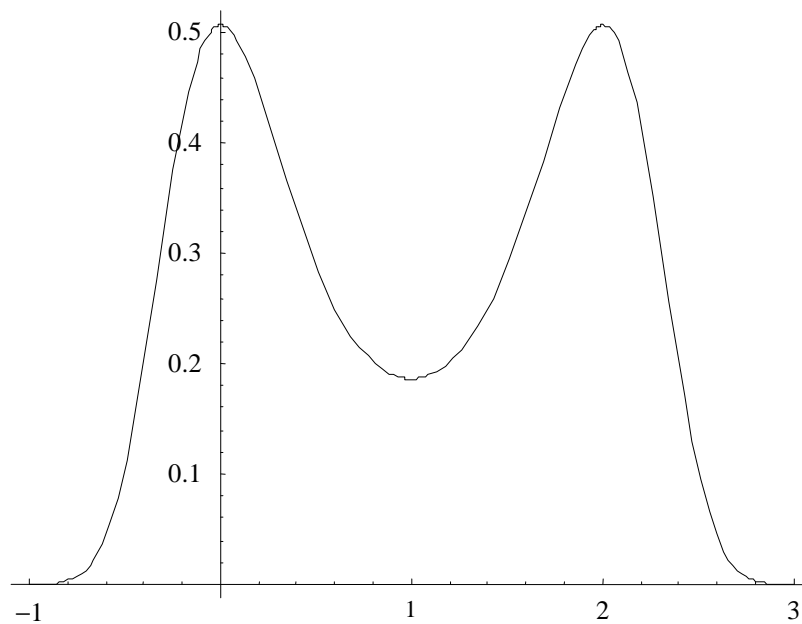


Fig. 2

Source: own research.

Modal values are in points:

$$Mo_1 = 0,$$

$$Mo_2 = 2,$$

while expected value  $E(X) = 1$  is in this case *unexpected*, since the probability mass is grouped around modal points. It can be seen that for parameters  $\alpha, \beta$  close to each other, Weber distribution is similar to that of normal distribution, which can be applied in the case of non-uniform samples, where two points of probability mass grouping can be observed, or in the case of discrimination problems. Weber distribution is characterized by uniformity property.

If

$$c \neq 0,$$

$$X \sim W(\alpha, \beta, \gamma),$$

then

$$cX \sim \frac{1}{|c|} \frac{z\left(\frac{\alpha}{c^2}, \frac{\beta}{c^4}\right)}{z(\alpha, \beta)} W\left(\frac{\alpha}{c^2}, \frac{\beta}{c^4}, c\gamma\right).$$

Proof for the above can be found in (A. Wilkowski (2008)).

In conclusion, it must be noted that with the increased number of modals, for analogous distribution patterns, determining the moments will pose increasing difficulties.

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