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REMARKS ON WIGNER'S SEMICIRCLE LAW

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Abstract. Ejsmont [2014] has shown that families of free Meixner distributions can be characterized by the conditional moments of polynomial functions of degree 3. In this paper, we will give other characterizations of the free normal distribution which are formulated in a similar spirit.

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1. Free probability, free cumulants, conditional expectation

We assume that our probability space is a von Neumann algebra A with a normal faithful tracial state $\tau : A \to C$ i.e., $\tau(\cdot)$ is linear, continuous in weak* topology, $\tau(XY) = \tau(YX)$, $\tau(1) = 1$, $\tau(XX^*) \geq 0$ and $\tau(XX^*) = 0$ implies $X = 0$ for all $X, Y \in A$. A (noncommutative) random variable V is a self-adjoint (i.e. $X = X^*$) element of A. The *-distribution μ of a self-adjoint element $X \in A$ is a probabilistic measure on R such that for all $n \geq 0$

$$\tau(X^n) = \int_R x^n d\mu(x).$$

Let $C\langle X_1, \ldots, X_n \rangle$ denote the non-commutative ring of polynomials in variables $X_1, \ldots, X_n$. The free cumulants are the k-linear maps $R_k : C\langle X_1, \ldots, X_k \rangle \to C$ defined by the recursive formula (connecting them with mixed moments)

$$\tau(X_1 X_2 \ldots X_n) = \sum_{\nu \in NC(n)} R_\nu(X_1, X_2, \ldots, X_n), \qquad (1)$$

where

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$$R_{\nu}(X_1, X_2, \ldots, X_n) := \Pi_{B\in\nu} R_{|B|}(X_i : i \in B) \tag{2}$$

and $NC(n)$ is the set of all non-crossing partitions of $\{1, 2, \ldots, n\}$ (see: [Nica, Speicher 2006; Speicher 1997]). Sometimes we will write $R_k(X) = R_k(X, \ldots, X)$.

Definition 1. *X is a free normal (Wigner's semicircle law) distribution if $R_k(X) = 0$ for $k > 2$. The Wigner semicircle distribution, is named after the physicist Eugene Wigner. The standardized, i.e. with mean zero and variance one, has density*

$$\frac{\sqrt{4-x^2}}{2\pi} 1_{[-2,2]}(x).$$

If $B \subset A$ is a von Neumann subalgebra and *A* has a trace τ, then there exists a unique conditional expectation from *A* to *B* with respect to τ, which we denote by $\tau(\cdot \mid B)$. This map is weakly continuous, completely positive, identity preserving, contraction and it is characterized by the property that, for any $X \in A$, $\tau(XY) = \tau(\tau(X \mid B)Y)$ for any $Y \in B$ (see: [Biane 1998; Takesaki 1972]). For fixed $X \in A$ by $\tau(\cdot \mid X)$ we denote the conditional expectation corresponding to the von Neumann algebra *B* generated by *X* and *I*. The following lemma has been proven in [Bożejko, Bryc 2006].

Lemma 1. *Let W be a (self-adjoint) element of the von Neumann algebra A, generated by a self-adjoint $V \in A$. If for all $n \geq 1$ we have $\tau(UV^n) = \tau(WV^n)$, then*

$$\tau(U \mid V) = W. \tag{3}$$

We introduce the notation

- $NC(n)$ is the set of all non-crossing partitions of $\{1, 2, \ldots, n\}$,
- $NC^k(m)$ is the set of all non-crossing partitions of $\{1, 2, \ldots, m\}$ (where $m \geq k \geq 1$) which have first k elements in the same block (see more: [Ejsmont 2014]).

Let *Z* be the self-adjoint element of the von Neumann algebra *A* from the above lemma. We define $c_n^k = c_n^k(Z) = \sum_{\nu \in NC^k(n+k)} R_{\nu}(Z)$ and the following functions (power series):

$$C^k(z) = \sum_{n=0}^{\infty} c_n^k z^{k+n}, \text{ where } k \geq 1 \tag{4}$$

for sufficiently small $|z| < \varepsilon$ and $z \in C$. The following lemma can be found in [Ejsmont 2014].

Lemma 2. *Let Z be a (self-adjoint) element of the von Neumann algebra A, then*

$$C^{(k)}(z) = M(z)C^{(k+1)}(z) + R_k(Z)z^k M(z), \tag{5}$$

where $k \geq 1$.

Example 1. For $k = 1$, we get:

$$C^{(1)}(z) = M(z) - 1 = M(z)C^{(2)}(z) + R_1(Z)zM(z). \tag{6}$$

In particular, we have the coefficients of the power series $1/M(z)$ (Maclaurin series):

$$\frac{1}{M(z)} = 1 - C^{(2)}(z) - R_1(Z)z \tag{7}$$

for sufficiently small $|z|$.

Similarly, by putting $k = 2$, we obtain:

$$C^{(2)}(z) = M(z)C^{(3)}(z) + R_2(Z)z^2 M(z). \tag{8}$$

Finally, we introduce moment generating function M_X of a random variable X by

$$M_X(z) = \sum_{n=0}^{\infty} \tau\left(X^n\right) z^n. \tag{9}$$

Now we present Lemma 4.1 of Bożejko and Bryc [2006], which will be used in the proof of the main theorem.

Lemma 3. *Suppose that X, Y are free, self-adjoint then X, Y have Wigner's semicircle law if and only if the moment generating function* $M(z)$ *for* $X + Y$ *satisfies the following quadratic equation*

$$2z^2 M^2(z) - M(z) + 1 = 0. \tag{10}$$

2. Characterization theorem

The main result of this paper is the following characterization of Wigner's semicircle law.

Theorem 1. *Suppose that X, Y are free, self-adjoint, non-degenerate, centered* $(\tau(X)=\tau(Y)=0)$ *and* $\tau(X^2)=\tau(Y^2)=1$ *which have the same distribution. Additionally, we assume that* $R_3(X)=R_3(Y)=0$. *Then X and Y have Wigner's semicircle law if and only if*

$$\tau\Big((X-Y)(X+Y)^2(X-Y)\,|\,(X+Y)\Big)=4I\,. \tag{11}$$

Proof. $\Leftarrow$: Let us suppose now that the equality (11) holds. Multiplying (11) by $(Y+Y)^n$ for $n\geq 0$ and applying $\tau(\cdot)$ we obtain

$$\tau((X-Y)(X+Y)^2(X-Y)(X+Y)^n)=$$

$$\sum_{\nu\in NC(n+4)} R_\nu(X-Y,X+Y,X+Y,X-Y,\underbrace{X+Y,X+Y,\ldots,X+Y}_{n-times})=$$

$$4\tau\Big((X+Y)^n\Big).$$

This follows from the folowing consideration. Let us look more closely at the second sum from the last equation. We have that either the first and the fourth elements are in different blocks, or they are in the same block. In the first case, the second sum (from the last equation) vanishes because we have

$$R_k(X-Y,X+Y,X+Y,\ldots,X+Y)=R_k(X)-R_k(Y)=0\,. \tag{12}$$

On the other hand, if they are in the same block, the sum disappears if the first or third element are in the same block separately because we have that $\tau(X+Y)=0$. So, we have

$$\tau\Big((X-Y)(X+Y)^2(X-Y)(X+Y)^n\Big)=$$

$$\sum_{\nu\in NC^4(n+4)} R_\nu(X+Y)+2\sum_{\nu\in NC^2(n+2)} R_\nu(X+Y)=4\tau\Big((X+Y)^n\Big). \tag{13}$$

This equation is equivalent to

$$4z^4M(z)=C^4(z)+2z^2C^2(z)\,. \tag{14}$$

Using Lemma 2 for $k=1,2,3$ we obtain equation

$$R_3(X+Y)z^3M^3(z)=(M(z)-1-2z^2M^2(z))(1+2z^2M^2(z)). \qquad (15)$$

Thus, if $R_3(X+Y)=0,$ then we have found two solutions

$$1+2z^2M^2(z)=0 \qquad (16)$$

or

$$M(z)-1-2z^2M^2(z)=0 \qquad (17)$$

but the first solution does not correspond to the probability measure. Part $M(z)-1-2z^2M^2(z)$ corresponds to the moment generating function of Wigner's semicircle law, which by **Lemma 3** implies statement. $\Rightarrow$: Suppose that *X* and *Y* have Wigner's semicircle law. Then we have

$$\sum_{\nu\in NC^4(n+4)} R_\nu\left(X+Y\right)=0.$$

So, from $R_2(X+Y)=2$ we see

$$\tau\Big((X-Y)(X+Y)^2(X-Y)(X+Y)^n\Big)=$$
$$2\sum_{\nu\in NC^2(n+2)} R_\nu(X+Y)=4\sum_{\nu\in NC(n)} R_\nu(X+Y)=4\tau((X+Y)^n). \qquad (18)$$

Now **Lemma 1** implies equation (11).

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