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SYSTEMS OF LINEAR EQUATIONS AND REDUCED MATRIX IN A LINEAR ALGEBRA COURSE FOR ECONOMICS STUDIES

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Abstract. The article presents two elements of the concept of a linear algebra lecture for economics studies. It attempts to demonstrate the significant role of ordering of the lectured content – with a focus on starting the lecture with systems of linear equations, and shows the considerable benefits of introducing the concept of the reduced row echelon form of matrix as one of the most useful concepts of linear algebra.

Keywords: linear algebra, reduced matrix, linear equations, economic studies, standards of teachings.

1. Introduction

The last few years have been a period of significant change in the teaching of mathematics and quantitative research methods at universities of economics (and all the others) in Poland. The reason lies in two main, mutually reinforcing factors – a significant reduction in the level of mathematics matriculation requirements and, simultaneously, decreasing the number of hours devoted to lectures on quantitative methods during studies. Attempts to clear up the confusion caused by these changes by adopting in 2005, the new "Law on Higher Education"[1] (standardization of names and curriculum content while reducing – by roughly one-third – the total number of hours devoted to lectures), did not produce the expected positive effects. As a result, from 1 October 2011, an amendment to the said Act entered into

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[1] Act "Higher Education Act" of 27 July 2005, Dz.U. (Journal of Laws) No. 164, item. 365.

force[2], which will also not cause a radical improvement of the situation. The most important change in the amendment relates to more freedom to shape the curriculum for some (by definition more prestigious) universities. The effects of the current six years of educational standards on the level of mathematics teaching in universities of economics are treated in the paper of Maciuk (2011). An analysis of teaching standards introduced in 2005, can be found in an article by Łyko (2007). In his article Dniestrzański (2011), conducted a discussion on the impact of changes in the education market in recent years, and the problems with the choice of field of study for high school graduates who are going to be educated in mathematical economics in a broad sense.

A thorough analysis of the effects of these seminal moments is not the subject of the article, but for illustration an example should be provided. In finance and accounting majors, most universities of economics in the 1990s devoted to mathematics about 120 hours of classes. Currently it happens, that to realize the curriculum (in accordance with the minimum requirements resulting from education standards) the lecturers have little more than 30 hours. This also occurs in a situation when secondary school graduates have much less knowledge than those of 20 years ago. This state of affairs makes the manner and the order of presenting the material more and more important in the teaching process. Out of necessity, only a small part of the time allowed to the lecturer is devoted to algebraic math course. The paper presents the elements of the concept of linear algebra lecture boiled down to two rules:

- at the beginning – systems of linear equations,
- reducing matrix as one of the basic tools.

2. Education content in linear algebra

Linear algebra is an important part of the course of mathematics for all economics majors in Polish universities. Table 1 shows the main (by, until recently, current educational standards) elements of the area of mathematics included in the program of activities.

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[2] Law Act – Law on Higher Education, the Law on Academic Degrees and Title and Degrees and Title in the field of art and some other acts dated 18 March 2011, Dz.U. (Journal of Laws) No. 84, item. 455.

Table 1. The main topics of linear algebra courses taught on economics majors

1	Systems of linear equations
2	Matrix calculus
3	Linear spaces
4	Determinants
5	Linear transformations
6	Inverse matrix

Source: own elaboration.

For the major of computer science and econometrics in ministerial education standards, there are also issues related to the linear and quadratic forms. In the case of most majors (logistics, economics, management, finance and accounting) training content related to linear algebra is restricted to a few slogans or (as for logistics) to one notion – *elements of linear algebra*. From my professional experience and the experience of others, I know that mostly the realization of the algebra curriculum (after drastic restrictions in the number of hours provided for quantitative subjects) comes down to the factors listed in Table 1, or a little beyond this framework.

Table 2 shows the content of education course for mathematics (for majors of computer science and econometrics, for which there are separate objects – mathematical analysis and linear algebra – the content of education for linear algebra was given) with respect to the algebraic part.

Even a cursory analysis of the curricula content given in Table 2 shows their considerable freedom. This applies especially to the level of granularity. These standards (except for, maybe, the majors of computer science and econometrics) contain substantially similar elements, but what, for example, for the major of logistics is called "elements of linear algebra", for the economics major is slightly more developed. The layout of Table 2 (aimed at highlighting these observations) shows the contents of the curricula of each major from most to least granular.

Table 2. The contents of curriculum with respect to linear algebra

Major (subject)	Content of curriculum
Computer science and econometrics (linear algebra)	Linear (vector) space. Linear independence of vectors. Base of the linear space. Vector coordinates. Transformations (homomorphisms) and linear matrix representation. Matrix algebra. Properties and classification matrices. Trace and rank of a matrix. The determinant of a square matrix. Inverse matrix. Systems of linear equations. Kronecker-Capelli theorem. Linear and quadratic forms. Canonical form of a quadratic form. Specificity and classification of quadratic forms – positive definite, negative definite
Finance and accounting (mathematics)	Matrix calculus. Determinants. Linear dependence and independence of vectors. Solving systems of linear equations and inequalities
Management (mathematics)	Vector and matrix calculus. Systems of equations and inequalities – examples in the field of management
Economics (mathematics)	Matrices, systems of linear equations, determinants
Logistics (mathematics)	Elements of linear algebra

Source: own elaboration based on educational standards of the Ministry of Science and Higher Education.

3. Systems of linear equations in selected textbooks

Systems of linear equations play a very important role in mathematics courses of all economics majors. The method of lecturing on this batch of material and its position in the course is often, in terms of the expected educational effects, not suitable. This is probably the result of underestimating the importance of this part of linear algebra in the understanding by the students of the next parts of the educational material. Problems with grasping this part of material have a definitely negative impact on the understanding of other subjects of linear algebra, which is a part of the mathematics course for all economics majors. This in turn causes problems in the reception of classes of some quantitative subjects such as econometrics, statistics and forecasting and simulations.

Table 3. Elements of contents of selected academic textbooks on linear algebra

Textbook	Chapter titles (subsections)
Textbook 1 – Banaś (2005)	1. Vector Space R^n 2. Matrices **3. Systems of linear equations** 4. Solving systems of linear equations and inequalities with the method of elementary operations 5. The use of systems of linear equations and inequalities in economic issues
Textbook 2 – Banaś (2005)	1. Vector space *n*-dimensional V_n 2. Convex sets 3. Matrices and operations on them 4. The determinant of matrix 5. Inverse matrix 6. **Systems of linear equations**
Textbook 3 – Piwecka--Staryszak (Ed.) (2004)	1. Linear space and metric space 2. Subsets of n-dimensional space 3. Distance subsets of n-dimensional space 4. Matrix algebra 5. Inverse matrix 6. Determinants 7. **Systems of linear equations** 8. Dimension and linear space base 9. Convex sets 10. Systems of linear inequalities and their applications 11. Diagonal form of a linear transformation matrix 12. Quadratic forms
Textbook 4 – Bednarski (2004)	1. Linear Space 2. Matrix and linear transformation 3. Determinant of matrix 4. Inverse matrix 5. **Solving systems of linear equations**
Textbook 5 – Antoniewicz, Misztal (2007)	1. **Solving systems of linear equations by elimination** 2. Space matrix. The concept of a vector space and subspace Scalar product, orthogonality 3. Linear envelope of set of vectors. Linear dependence and independence. The basis 4. Linear transformations 5. Exterior product (oblique). Determinant and oriented volume 6. Linear equations. Inverse matrix. Rank of the matrix. Transition matrix 7. Vector product. Line. Hyperplane. Sphere 8. Eigenvectors and eigenvalues of linear transformation. Orthogonal matrices 9. Linear forms. Quadratic forms 10. Hyperspace of the second degree in space

Source: own elaboration.

In most mathematics textbooks for economics majors, solving systems of linear equations is at the end of the section dealing with the problems of linear algebra. To illustrate this, Table 3 shows the elements of the contents of selected academic textbooks.

Table 3 shows that in four out of five cases (of those analyzed) systems of linear equations appear later than the matrix calculus and, in three out of five cases, later than the study of linear independence of vector systems. Also, determining the inverse matrix (in four out of five cases) comes earlier. If, for the elements of linear algebra, one could use 2-3 times more time than is done currently, such a scheme of teaching may be acceptable, and one could even find arguments that it is didactically preferred, however, in the current state of affairs this is difficult to accept. Starting a linear algebra course with a general theory of solving systems of linear equations has a double meaning. Firstly, it is useful in successive parts of the course. Secondly, the technique of solving (the reduction of the matrix) can be directly (and independently of solving systems of equations) used as a tool, for example when examining the linear independence of vector systems or determining the invertible matrix.

4. Reduced matrix

The concept of the reduced matrix is very useful and relatively easily absorbed by the students. Mastering the matrix reduction techniques allows it to be used in a number of algebraic problems. Below is the definition of an example of the reduced matrix.[3]

Definition.

Row Reduced matrix (or briefly, **reduced matrix**) is a matrix in which for each non-zero row there is at least one element, which is the only non-zero element in its column.

Example 1. Let

$$A=\begin{pmatrix} 0 & 0 & 3 & 0 \\ 0 & \boxed{8} & 0 & 4 \\ \boxed{7} & 0 & 1 & 0 \end{pmatrix}, B=\begin{pmatrix} \boxed{2} & 4 & 0 & 0 \\ 0 & 8 & 0 & \boxed{4} \\ 0 & 0 & \boxed{-5} & 0 \end{pmatrix}, C=\begin{pmatrix} -5 & 0 & 2 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \\ 7 & \boxed{3} & 1 & 2 & 0 \end{pmatrix}$$

[3] Definition 1 and Example 1 is one of the first elements of the lecture of the algebraic part of the course in mathematics introduced by the author.

Matrices B and C are reduced, and A is not a reduced matrix. Elements referred to in definition 1 (which are the only non-zero elements in their columns) have been marked. Note that in matrix A in the second row item 4 could also be marked – it is also the only non-zero element in its column.

5. Applications of reduced row echelon form of matrix

5.1. Systems of linear equations

Solving systems of linear equations by reducing the augmented matrix is a natural and easily understood technique. It can be used even without a precise definition of the concept of the reduced matrix.

Example 2. Let us consider the system of equations

$$\begin{cases} 3x + 2y - 6z = 3 \\ x - 3y - 2z = -10 \\ -x + y + 2z = 4 \end{cases}.$$

Presentation to students of the following operations:

$$\left(\begin{array}{ccc|c} 3 & 2 & -6 & 3 \\ \boxed{1} & -3 & -2 & -10 \\ -1 & 1 & 2 & 4 \end{array}\right) \begin{array}{c} w_1 - 3w_2 \\ \to \\ w_3 + w_2 \end{array} \left(\begin{array}{ccc|c} 0 & 11 & 0 & 33 \\ 1 & -3 & -2 & -10 \\ 0 & -2 & 0 & -6 \end{array}\right) \begin{array}{c} w_1 : 11 \\ \to \\ w_3 : 2 \end{array} \left(\begin{array}{ccc|c} 0 & \boxed{1} & 0 & 3 \\ 1 & -3 & -2 & -10 \\ 0 & 1 & 0 & 3 \end{array}\right) \begin{array}{c} \to \\ w_2 + 3w_1 \\ w_3 - w_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 3 \\ 1 & 0 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \begin{cases} \boxed{y} = 3 \\ \boxed{x} - 2z = -1 \end{cases}$$

and providing the general solution of the set $x = 2z - 1$, $y = 3$, $z \in R$ is accepted with understanding and interest.

Solving systems of equations by this method has an important advantage over the classical Gaussian elimination method or the Gauss-Jordan elimination. The use one of the latter methods in Example 2 would require replacing the variables – swapping columns of the augmented matrix or dividing the relevant row by a constant. In addition, the matrix reduction method allows for the easy control of the reduction process when we have certain expectations of the division of the unknown into variables and parameters (decision variables). Imagine, for example, that in the above example we obtain a general solution where the unknown *x* is a parameter. This is due to the one obvious limitation – the reducer[4] cannot be taken from the first column of the augmented matrix, the columns representing the unknown. The Gaussian elimination method in Example 2 will lead (if we do not swap the columns) to the general arrangement in which the parameter is the unknown *z*. It is not possible to obtain another form of general solution without having to swap columns.

The classification of systems of equations due to the number of solutions for the matrix reduction method is also simple and natural – a consistent system is undetermined if and only if during the reduction in at least one column a reducer will not appear.

One should also notice that the formulation in the theory of systems of linear equations for the unknowns, that are real numbers of the Kronecker--Capelli theorem, is by far a triumph of form over content. This is in fact a theorem in the following form: if $a \neq 0$, then the equation $0 = a$ is contradictory. In a linear algebra course, limited to the issues specified in Table 1, it is dubious even to introduce the concept of a matrix rank unless one would like, alongside discussing linear transformations, also to describe their images. Although even in this case, the concept of a rank of the matrix can be easily removed.

5.2. Linear independence of vectors

In solving systems of linear equations using the matrix reduction method, we inform the students that the rows of the augmented matrix that represent equations dependent (linearly) from other equations, disappear (are zeroed), especially since just before zeroing of a given row another row is rescaled (containing a reducer). In this way, the theory of linear space is a very simple way of displaying the intuition associated with the linear dependency of vectors. After providing the formal definition, one can relatively quickly move on to the following assertion.

Theorem 1. The system of linear space vectors is dependent if and only if during the matrix reduction formed of these vectors (the coordinates of one of the vectors form a single row of the matrix) at least one row will be zeroed.

[4] The reducer is a selected element of the matrix which is the "reduction tool". In Example 2, the reducer is, for example, in the first matrix (marked) element which is in the second row and the first column.

The thesis statement is generally regarded by students as natural and accepted without resistance, without proof. Furthermore, the essence of evidence in this case is exceptionally easy to produce. It is not without significance that the theorem allows for the much faster verification of linear relationship then by solving the appropriate system of equations.[5]

Example 3. To investigate the linear independence of the system of vectors $a = (-2, 3, -1, 1)$, $b = (5, 7, 0, 2)$, $c = (7, 4, 1, 1)$ one should just perform the following reduction:

$$\begin{pmatrix} -2 & 3 & -1 & \boxed{1} \\ 5 & 7 & 0 & 2 \\ 7 & 4 & 1 & 1 \end{pmatrix} \begin{matrix} \to \\ w_2 - 2w_1 \\ w_3 - w_1 \end{matrix} \begin{pmatrix} -2 & 3 & -1 & 1 \\ 9 & \boxed{1} & 2 & 0 \\ 9 & 1 & 2 & 0 \end{pmatrix} \begin{matrix} w_1 - 3w_2 \\ \to \\ w_3 - w_2 \end{matrix} \begin{pmatrix} -29 & 0 & -7 & 1 \\ 9 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and using Theorem 1 we find that these vectors are linearly dependent.

Testing the linear independence of the vectors by a reduction of the matrix has, in addition to the aforementioned simplicity and low labor intensity, the following advantages:

1. In the case of a dependent system it is easy to identify combination vectors of the tested system which give the zero vector – just follow the operations that led to the zero row.
2. The fact that the system containing the zero vector is dependent seems to be obvious.
3. The method is close to the definition of linear independence – it is easy to notice that the zero vector obtained during reduction is a combination of vectors of the given system.

5.3. Determinants

The components of matrix reduction may be used in the practice of calculating square matrices determinants, especially if the students have already mastered the technique of reduction. Take the following example.

[5] In the case of direct use of the definition of linear independence as a system, the combination which gives only the zero vector is a trivial combination, we must first create a system of equations, and then solve it.

Example 4. (The arrow indicates the column for which the Laplace transform is used)

$$\begin{vmatrix} 3 & 5 & \boxed{1} & 0 \\ 7 & 0 & 2 & -3 \\ 2 & 2 & 4 & 7 \\ -2 & 1 & 0 & 7 \end{vmatrix} \begin{matrix} w_2 - 2w_1 \\ = \\ w_3 - 4w_1 \end{matrix} \begin{vmatrix} 3 & 5 & \overset{\downarrow}{1} & 0 \\ 1 & -10 & 0 & -3 \\ -10 & -18 & 0 & 7 \\ -2 & 1 & 0 & 7 \end{vmatrix} =$$

$$= 1 \cdot (-1)^{1+3} \begin{vmatrix} 1 & -10 & -3 \\ -10 & -18 & 7 \\ -2 & \boxed{1} & 7 \end{vmatrix} \begin{matrix} w_1 + 10w_3 \\ = \\ w_2 + 18w_3 \end{matrix}$$

$$= 1 \cdot \begin{vmatrix} -19 & \overset{\downarrow}{0} & 67 \\ -46 & 0 & 133 \\ -2 & 1 & 7 \end{vmatrix} = 1 \cdot (-1)^{3+2} \begin{vmatrix} -19 & 67 \\ -46 & 133 \end{vmatrix} =$$

$$= (-1)(-19 \cdot 133 - 67 \cdot (-46)) = -555.$$

The possibility of using the elements of matrix reduction for the calculation of the determinants is clear and is to be found in practically every linear algebra course. However, the use of the matrix reduction by students prior to solving systems of linear equations and testing linear independence of vectors, causes that, when reaching the notion of determinants (or successive elements of linear algebra where you can also use this technique), the ability is already known, mature and does not cause any difficulties.

5.4. Other applications

The operations performed in Example 3 allow us to provide a sample basis (and dimension) of linear subspace generated by the vectors a = (–2, 3, –1, 1), b = (5, 7, 0, 2), c = (7, 4, 1, 1). This consists, of course, of vectors created from the unreduced lines the reduced matrix, i.e. (–29, 0, –7, 1) and (9, 1, 2, 0). Therefore, we have another example of the technique, and additional applications received somewhat "for free". Examining the linear independence of vectors *a*, *b*, *c* by solving the appropriate system of linear equations can, indeed, determine the dimension of the subspace lin{*a*, *b*, *c*), but it does not indicate the basis. A reduction of the matrix can also be successfully used, for example, to determine the inverse matrix or establishing the image of linear transformation.

6. Conclusions

Limiting the number of hours of classes for quantitative subjects in economics majors and limiting mathematics education in high school enforces the concept of fundamental change in providing lectures and exercises in mathematics. The elements of linear algebra offered to students (more or less boiled down to those listed in Table 1) can be greatly enhanced using the concept of reduced matrix and the matrix reduction techniques. The presented examples of its applications have been used and successfully tested in practice in mathematics courses in various economics related studies at the Wroclaw University of Economics. The degree of mastery achieved by the students, that has been proved with regular tests and revised during final exams, shows that the concept is at least worthy of consideration. The specificity of economics studies in which mathematics is an auxiliary subject, puts great pressure on students to master the skill in the use of mathematical tools. The elements of the lecture concept of the algebraic part of the mathematics course can help to achieve this effect.

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