D I D A C T I C S O F M A T H E M A T I C S

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EXAMPLES OF METRICS FOR ECONOMISTS

Abstract. For teachers metrics are a perfect tool to practice abstract thinking and spatial imagination. For economists metrics facilitate describing and making models of economic processes. Thus it is worthwhile to present, especially to students taking basic math at economy studies, a few exercises of metric constructions describing certain cases, like drawing spheres in these metrics. In this article such metrics are proposed together with methods of their creation. They are simple and can be treated as examples of exercises for students. At the same time they show how to construct models suitable for more complicated situations – e.g. how to create metrics in which distance would correspond to the cost of covering such distance (or time required for that purpose). For students, future economists, having such knowledge will facilitate constructing similar models, which can be useful for their work.

Key words: graph-based metric, metrics in didactics, models of economic processes.

1. Discrete metric

For a student drawing a sphere in the Euclides' metric (d_2 metric) is an easy and intuitively comprehensive task. Sphere is a set of points x, whose distance to a center p is not bigger than a given r. So: $K(p, r) = \{x: d(x, p) < r\}$. Drawing spheres in the City metric (d_1) or Chebyschev's metric (d_∞) is not a problem either. It is enough to disqualify the points whose distance from p is bigger than r.

An interesting and often problematic task is to draw a sphere in the discrete metric. Discrete metric is the one which takes a finite number of values. The easiest one is a function $d(x, y) = 0$, where $x = y$ and $d(x, y) = 1$, where $x \neq y$. When $r > 1$, the sphere in this metric is the whole space, regardless of the center p location. However, even if students are able to come to this conclusion, they think that it cannot be true as it is against their intuition. A sphere must be *finite* – they say – sphere cannot be the whole space!

This is when the new opportunities appear. What does it mean that the set is unlimited? Doesn't it have a limit? What is a limit of a set? Let's assume that a limited set is a set which is closed in a sphere of a finite radious. Is line in that case a limited set? Well, that depends in what metric. In the discrete metric it is! Through asking such questions and discussing different answers on various examples a teacher can easily and in a natural way engage students in mathematical discussions and teach them abstract thinking. Another important aspect is that it takes only 5-10 minutes to achieve this goal.

A more complex example of the discrete metric is the metric which reflects the policy of telephone fees.

In the discrete metric each set is open and closed at the same time, thus spheres are also open-closed sets. For students a very interesting example of such set is a sphere in the *Telephone fees metric*. In the telephone communications there are various fee levels used depending on the location of the people talking. If we are calling a person from the same city or district we will pay less than when talking to someone from far away. We will be charged even more if we make an international call. Let's contruct a metric which reflects this situation.

$$d(x,y)=\begin{cases}0 & \text{for } x=y,\\ 1 & \text{for } x\neq y \wedge x,y\in M,\\ 2 & \text{for } x,y\notin M \wedge x,y\in W,\\ 3 & \text{for } x,y\notin W \wedge x,y\in K,\\ 4 & \text{for } x,y\notin K,\end{cases}$$

where M, W, K are any nonempty sets (where $M \subset W \subset K$) which contain city, district, country and x, y are locations of callers.

And how do spheres look like in this metric? A sphere of radius 1 and center which is a location of a certain caller in any city is a set of all locations (addresses of callers) from that city (not the city itself!). And a sphere of the same center but a radius 2? These will be locations of all callers from the whole district.

Another benefit from this example is making the students realize that it is quite easy to construct a metric in which distance corresponds to the costs of covering such a distance.

2. Metric with an obstacle

An easy and interesting example of a case which can be presented with the use of metrics is modelling disturbance generated by the existence of an obstacle, which divides a certain set into two parts which have only one common point. The obstacle can be a river with a bridge or a border with just one border crossing. Such situation is reflected by the following metric:

$$d_c(x,y) = \begin{cases} d(x,y) & \text{for } x, y \in A \vee x, y \in B, \\ d(x,c) + d(c,y) & \text{for } (x \in A \wedge y \in B) \vee (x \in B \wedge y \in A), \end{cases}$$

where $A \cap B = \{c\}$ (or $A \cap B = \{\varnothing\}$ and $c \notin A \cup B$).

An interesting exercise for students is drawing spheres in this metric (Fig. 1).

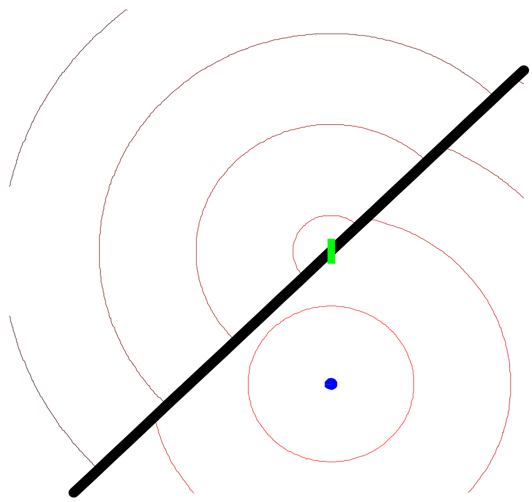

Fig. 1. Spheres in metric with obstacle and one crossing
Source: own elaboration.

3. Metric of two speeds

A minimum function makes it possible to construct metrics adequate for models with different speed movements. Thanks to vehicles we can cover distances faster, but only when going by well prepared routes. To be able to go faster we first need to get to such a road. Not always it is necessary, as many places we can reach on foot. However, if it is faster to get somewhere by car we need to first get to the closest point of a road. From there we take the car to the point closest to our destination and cover the rest of distance on foot again. So in this case the distance from the point *x* to point *y* is presented by the following equation:

$$d_s(x,y) = \min\left\{ d(x,y), \min_{p\in D} d(x,p) + \min_{q\in D} d(q,y) + \tfrac{1}{10} d(p,q) \right\},$$

where *D* is a route which enables to move at the speed 10 times faster than the standard (Fig. 2).

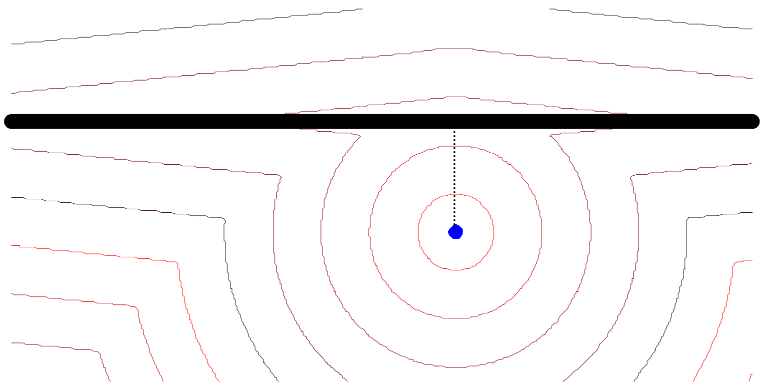

Fig. 2. Spheres in metric of *two speeds*

Source: own elaboration.

4. Graph-based metric

In the case of railway or airplane transportation a passenger cannot get on the train or plane at any route point, only in strictly defined locations (airports or train stations). While constructing metrics for such situation one may use the notion of a graph. A very interesting is the case in which there are shortcuts between locations. Thanks to these shortcuts spheres in a graph-based metric may be *non-coherent*.

Let's look at a graph which consists of two nodes *a* and *b* and an edge joining these points with corresponding *length p* (where $p > 0$). Let's assume that these two points (nodes) are included in a set, where metric *d* is described. If parameter *p* is significantly smaller than $d(a, b)$ than metric d' can be described by the equation:

$$d'(x,y) = \min\{d(x,y),\ d(x,a) + p + d(b,y),\ d(x,b) + p + d(a,y)\}.$$

The above equation can be easily interpreted. We have two forms of transport available and the second faster one can be used only directly between points *a* and *b*. A passenger can travel between *x* and *y* by the means of the first slower transport ($d(x, y)$) or use it to get to a transit point ($d(x, a)$ or $d(x, b)$). From there, again with the use of the faster transportation, he gets to his destination ($d(b, y)$ or $d(a, y)$). This equation describes the situation in which the passenger chooses the option which enables him to get to the final point faster. In this metric spheres have a characteristic form (Fig. 3).

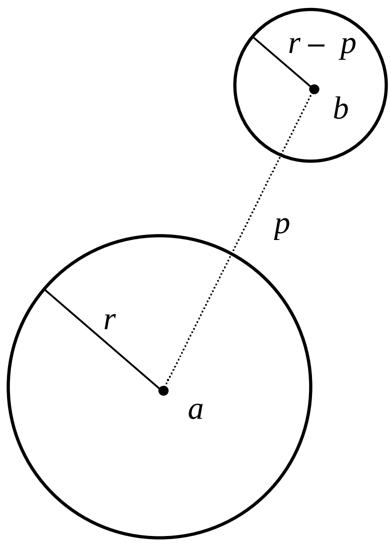

Fig. 3. Circle with the center in point *a* and radius $r > p$ in graph-based metric
Source: A. Maciuk (2004).

The metrics using more complicated graphs reflect various situations very well. In Fig. 4 we can see spheres in a metric based on a graph, wich consists of a few edges of p_i length joining a few various points b_i with a central point *a*. Such metric is described by the following equation:

$$d'(x, y) =$$
$$= \min\{d(x, y), \min_{i=1,\dots,n}\left(\min\{d(x, a) + p_i + d(b_i, y), d(x, b_i) + p_i + d(a, y)\}\right)\}.$$

In the example shown in Fig. 4 the node *a* corresponds to location of Warsaw airport, and points b_i to locations of regional airport, which operate flights to Warsaw. Parameters p_i were determinated in such a way that they reflect the duration of each flight.

The above mentioned examples do not cover the whole topic. Similar and more complex examples of metrics can be found in the books by C. Ponsard (1988), W. Pluta (1986) and W. Odyniec, W. Ślęzak (2003).

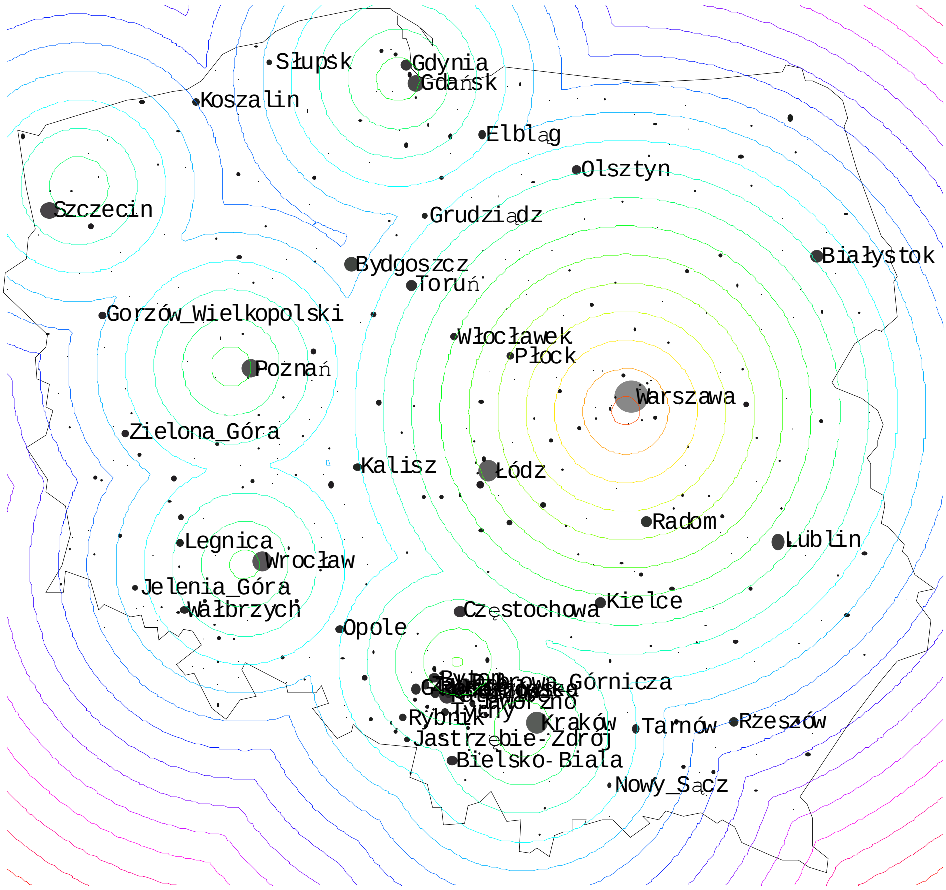

Fig. 4. Map presenting the time it takes to get to the Warsaw airport
from various places in Poland (spheres in graph-based metric)
Source: A. Maciuk (2004).

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