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A FEW EXAMPLES OF NON-STANDARD SOLUTIONS FOR THE CALCULUS OF PROBABILITY PROBLEMS

Abstract. The authors present and analyse untypical solutions to standard problems in calculus of probability at the secondary school level and a rudimentary course level in economic universities. They also discuss their influence on the understanding of the discussed issues among secondary school pupils and university students.

Key words: calculus of probability.

This article is an outcome of discussions conducted by us during the breaks while running preparatory courses for the prospective students in Wroclaw University of Economics. Exchanging our views on the level of preparation presented by the candidates, we realized that in solving some of the tasks in probability calculus they often use unusual and strange methods. The confirmation of it was a situation which took place in July 2005 during the entrance exams for the University of Economics. One of the given five tasks went as follows.

Task one (entrance exams 2005).There are fourteen white balls and six black ones in an urn. Calculate the probability of the event, where among five balls chosen at random exactly three are white.

It would seem that this is a standard task in the theory of combinations, for combinations and the use of the Newton’s symbol. However, while checking the papers, we realized that approximately 30% of the candidates attempted to solve the problem using so called *tree,* sketching five stages of drawing the five balls, and then choosing *paths* meeting the conditions of the task. What is interesting is that many of them obtained the correct solution using this way. The problem arose as to how to grade such solu-tions. From a formal viewpoint the task was solved correctly, but using a very poor method. Those approving of such a method should be asked immediately, what would a student do if the draw involved 50 balls out of 70? Commenting on this, Dr. Wojciech Rybicki said that *trees* are a plague of secondary schools, and some teachers do not teach any other methods since they see the use of *trees* as the way to avoid the theory of combinations, conditional probability, total probability, independence, and such likes constituting after all the core of a classic calculus of probability.

The above comment coincides with some of the theses proposed in this paper, since we would like to draw attention to a few tasks which in our opinion are artificially created only to force a pupil to apply a formula for total probability, or drawing a *tree*. We would like to stress clearly here that we are not opposed to the formula for total probability (this would be absurd!). Neither do we want to condemn the *trees* altogether, knowing that in many cases they are the only sensible method to describe a probabilistic situation. However, we are opposed to creating problems artificially only to serve specific methods of their solution, since we see this as non-didactic and non-scientific. In science we are finding solutions to problems, and not otherwise. This coincides with what Professor Antoni Smoluk says, that mathematics simplifies and finds the shortest way to the point. Anyone who complicates simple things and goes about in a roundabout way, practises anti-mathematics, which should be fought against.

Here is another example. In every calculus of probability textbook there can be found the following type of a task.

Task two. From an urn where there are three white balls and seven black ones, two balls are drawn, and then one is chosen. Calculate the probability that a white ball was drawn.

Obviously we can once (stressing: just once) allow a pupil to solve this task in a lazy way. Here is such a solution (Fig. 1):

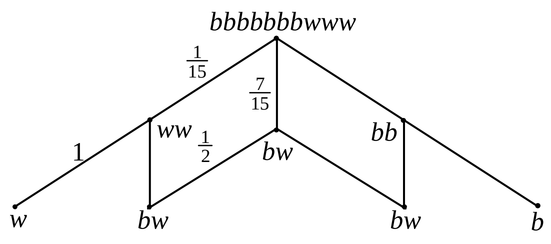

Fig. 1

$$P = \tfrac{1}{15}\cdot 1 + \tfrac{7}{15}\cdot\tfrac{1}{2} = \tfrac{1}{10}.$$

However, we would allow the pupil to use the above solution only to show him/her that the result is the same as if we drew that ball directly out of the ten balls. Because it cannot be different!!! In the final solution, in a random way, we chose one ball out of ten among which there were three white ones. The chance for a white ball cannot be other than 0.3, and if someone insists upon the *tree* we suggest solving the next task.

Task three. From the urn where there are three white balls and seven black ones, five balls are drawn, out of which three are chosen. Out of the three we draw one ball. Calculate the probability that a white ball was selected.

This probability is still 0.3. The task can be still complicated by saying that first the balls are placed at random in three urns, and then one urn is drawn, out of which three balls are extracted, and out of those one is chosen. Such a task cannot be in fact solved in a classic way, and in our opinion we can straight away explain it to a pupil so he/she can see the solution (result) at once.

Conclusion

If among n elements there are k differentiated ones, then **independently from the mechanism of drawing** them the probability of selecting the differentiated element equals $P = \frac{k}{n}$.

The above conclusion is obvious, and it provides a kind of comment on the classic definition of the calculus of probability.

And now we shall present two other tasks taken from the set of *matura* exams (K. Cegiełka, J. Przyjemski (1991)).

Task four. Out of a set of exam tasks consisting of 10 algebra problems, 9 geometry problems and 6 calculus of probability problems, one problem was selected at random and put aside without being looked at. Next a second problem was drawn. Calculate the degree of probability that the second one was related to algebra.

We can bet that every pupil (unfortunately also almost every teacher) will immediately start with drawing a *tree* (incidentally, such a tree is usually needlessly drawn with three branches, i.e. algebra, geometry and calculus of probability, instead of just two – algebra and non-algebra). And yet this task is identical with the task two, only it should be presented in the following way: Among 25 elements 3 are differentiated. We choose out of these 24 (because to put aside 1 means to select 24), and out of these 24 we choose 1. Calculate the probability of choosing the differentiated element.

The result cannot be other than 10/25, i.e. 2/5. Naturally, by using a *tree* we shall obtain the same result, but at this point we have started thinking about a situation where such tasks are related to seven mathematical disciplines, and first one task is put aside, then 3, then 5 more, and finally one is drawn. We wish the fans of *trees* good luck.

At this point it is not difficult to convince the pupils that the thesis contained in the following task is obvious.

Task five (K. Cegiełka, J. Przyjemski (1991)). In an urn there are m tickets, among them k winning tickets ($m > 2$ and $k < m$). One ticket is taken out of the urn by the first person, and then out of the remaining ones a second ticket is drawn by the second person. Show that the probability of both people drawing a winning ticket is the same.

Obviously both these probabilities equal k/m. Every pupil will believe that a possible third, fourth and other persons have the same chance to win. Following this train of thought we shall propose now the following task.

Task six. Out of the set {1, 2, 3, 4, 5, 6, 7} we draw in turn four numbers and by arranging them in order of draw we create out of them a four figure number. Calculate the probability of obtaining in this way an even number.

Unfortunately, there will be probably a difficulty in connecting this task with the previous ones. Many pupils will certainly ask whether the draw involves returning or not. We are convinced that the answer *this has no importance in this task* will surprise not just the pupils. Obviously many of them will immediately answer that if we draw with returning then the probability of drawing for the fourth time an even number (and this is after all what this task is about), is the same as at the first time, i.e. 3/7. And if we draw without returning? Also 3/7. Because there are 7 elements, 3 differentiated ones, we put aside one, then another, then the third one, and then we draw one. We can also use another argument: at the beginning of the experiment every number has identical odds of being drawn as the fourth one, so the odds for an even number are 3/7.

Here is another example of a task where we can drop tedious calculations.

Task seven. In a lottery there are 7 winning numbers, 10 losing ones and 2 which give us the right to draw again. Calculate the probability that when buying one ticket we shall win (assuming that we use the right to possible further draws).

We can also allow pupils to solve this task by using a classic *tree*. Its 3 branches, signifying situations in which we win, give us probability

$P_1 = \frac{7}{19}$, $P_2 = \frac{2}{19} \cdot \frac{7}{18}$, $P_3 = \frac{2}{19} \cdot \frac{1}{18} \cdot \frac{7}{17}$, ($P_i$ means probability of winning in the i-th draw). Hence the probability of winning equals

$$P = P_1 + P_2 + P_3 = \frac{7}{17}.$$

However, we would allow pupils a *classic* solution only so that they could see that this solution is trivial. If they could not see it, we would give them homework in which there would be 50 tickets giving an additional chance. After such a modification usually one can hear the right answer: such tickets have no importance, and the probability remains the same as if they did not exist! And for those who still are not convinced we suggest the following picture explanation of the phenomenon. Let us imagine that there is an urn, winning tickets are white balls, the losing ones are black, and the ones giving another chance to win are like soap bubbles that burst and disappear when you draw them. And there are very many of such *bubbles*. We put the hand inside such an urn and try to extract a ball. Definitely the first few (or even several) will burst. But finally we shall get one which will not. Is it important how many times it takes? No!!! The experiment will end when we get one out of the 17, which can be taken. What is the probability of it being white? 7/17 and it cannot be different.

Let us note that the last task is a recipe for an attractive lottery. We know from the experience that this example also allows students to grasp the nature of the phenomenon. Let us imagine that we organize a lottery. We prepare 500 tickets at 1 złoty each, out of which some win (let it be zł 10 to 50), the rest loses, but everything is calculated in such a way that if we sell all the tickets we will make some money. Then we add tickets to our lottery, for example 3000, which cost 1 złoty and win also 1 złoty. These bursting balls are the tickets giving another chance, because in this situation every player asks for another ticket. The students feel that these tickets have no importance, that the odds to win will not change, and we will make money anyway. However these tickets have enormous psychological significance. Thanks to them our lottery will be very attractive, because many participants at a cost of 1 złoty will have a chance of a multiple draw, which will *sweeten* the fact that in the end they will probably lose anyway. So perhaps they will buy another ticket.

In the end, as a curiosity, we would like to include here the solution to the task four given in the book from which we took it (K. Cegiełka, J. Przyjemski (1991)). We present this solution as an example not to follow. Especially that in the above mentioned collection of tasks (and all the others we looked through) there is not a single word about other methods of solving such problems.

Out of the set of 25 *subjects we draw without returning* 2 *and we take interest in their order. We assume as a set of elementary events … the set of all diverse value two-term sequences of terms belonging to a set of* 25 *subjects.*

Therefore $\overline{\overline{\Omega}} = 25 \cdot 24 = 600$. *We accept all elementary events as equally probable. We mark as B an event where at the second go a subject from algebra is drawn, C – where both subjects are drawn from algebra, D – where at the first go a subject from geometry or calculus of probability is drawn and at the second go one from algebra.*

We have

$$\overline{\overline{B}} = \overline{\overline{C}} + \overline{\overline{D}} = 10 \cdot 9 + (9 + 6) \cdot 10 = 240$$

and finally $P(B) = \frac{\overline{\overline{B}}}{\overline{\overline{\Omega}}} = \frac{240}{600} = \frac{2}{5}$.

The problems of teaching calculus of probability (at every level of education) are connected to a large degree with the lack of absolute precision in this mathematical discipline. As we know mathematics is based on such basis as axioms, i.e. theorems accepted without argument. On the basis of axioms, set rules of building sentences and the rules of reasoning these formulas, for which a formal argument exists, become mathematical theorems. Naturally, we have known since the times of Kurt Godel that in every such formal system always exist unsolvable tasks, the ones that can be neither proved nor disproved.

The problem with calculus of probability comes from the fact that we always ascribe to elementary events defined probabilities in an arbitrary (or even better said *wishful*) way. Only after constructing the space of elementary events we can assess the probability of particular events clearly and according to the set premises. The construction of the space of elementary events reminds of the situation where every time (with the assessment of probability of events) we construct a new formal system. The assumptions made determine the conclusions. Such situation is most often the source of ambiguities and misunderstandings. It is also the reason of so many so-called paradoxes existing in the theory of probability. Perhaps the most famous of them all is the paradox of Villa Sorbelloni. Departing somewhat from the original we can present it in a following way: in three urns there are three balls – one in each. The two balls are white and one is black. We draw one urn and extract one ball from it. We are interested in the probability of extracting the black ball. We do not have to convince anyone that such probability amounts to $P = 1/3$. We made an extraction and we hold a ball in our hand without looking at its colour. Not depending on the colour of the drawn ball in at least one urn there is still a white ball. Standing with the extracted ball, and still not knowing its colour, we can see that one of the urns falls apart and a white ball rolls out of it. Does at this moment the probability of the event of us holding a white ball change? Is it $P = 1/2$ at that very moment? Or did the destruction of one of the urns not change the probability that interests us and it is still $P = 1/3$? Any group of mathematicians will immediately split into two sides. Not to mention *mathematical laymen*. It is enough to recall that a heated discussion on this subject once almost brought down to a closure a conference of biologists (Villa Sorbelloni, 1966). We shall not conduct a discussion here, or give arguments in favour of one or the other view. Especially since our opinions differ.

We have recalled here that famous paradox only to alert even more those who teach the theory of probability. This theory is sufficiently ambiguous in itself so that at least where it is possible we should avoid complications, creating artificial problems. The pupils and students hungry for knowledge will notice by themselves, sooner or later, that not everything in the theory of probability is unequivocal, and often in order to solve a problem mere mathematical skills are not enough, it is also necessary to have some general knowledge of the world. Someone who has not seen a tossed coin may doubt the odds of it falling tails up.

Final conclusions

The purpose of this article is to draw attention to some (in our opinion) shortcomings of teaching probability calculus in secondary schools. Perhaps we would not put forward our opinions if we were not aware of the serious damage caused by some *didactic techniques* practised for many years at this level of education. To prove how deeply they are entrenched we can confirm that we were unable to find any textbook or a set of tasks, which would mention at least the existence of other ways of looking at the issues of probability calculus mentioned in this article. Not to elaborate too much on the potential consequences of this state of things for the pupils (which are obvious enough even during correcting entrance exam papers), we would like to suggest that perhaps university courses on some subjects provide a chance to redeem this situation.

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