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ON CHANGING MONEY AND THE BIRTHDAY PARADOX

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Abstract. The paper deals with two problems of discrete probability theory that can be included to a curriculum in mathematics or statistics at universities of economics. The first one addresses the question that is interesting for economists: in how many ways one can change a certain amount of money given a fixed set of denominations. One can answer the question using either theoretical or software tools (such as Matlab or Mathematica). The other problem deals with the birthday paradox and its generalization. It is an example when intuition fails.

Keywords: birthday paradox, generating function, partition.

1. First, we will discuss several facts connected with partitions of natural numbers. A number of all partitions of a natural number n is denoted by p_n. Partitions that have different order of elements are considered to be equal. Hence, it is easy to discover that a sequence (p_n) with $n \in N$ increases at a fast pace because, for instance, p_{200} = 3972999029388, and p_{14031} has 127 decimal figures (Narkiewicz 1977).

An effective method to analyse sequences of numbers is to work with infinite power series of complex numbers that produce a given sequence, i.e., with generating functions. We consider them as formal series that may be added, multiplied, differentiated, etc. They are a good representation of their coefficients. Their convergence is not typically relevant (unless asymptotic analysis of coefficients is needed). In the case of partitions we denote a generating function by $P(z)$. We have

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \ , \quad z \in \mathrm{C}.$$

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It is known (Palka, Ruciński 1998) that if $|z| < 1,$ then

$$P(z) = \prod_{n=1}^{\infty} \left(1 - z^n\right)^{-1},$$

where the right-hand side product is absolutely and almost uniformly convergent. While comparing generating functions, it is easily shown that the number of partitions into distinct parts is equal to the number of partitions into odd parts. The respective generating functions are given by:

$$R(z) = (1 + z)\left(1 + z^2\right)\ldots$$

The first parenthesis in the above product means that 1 appears in the partition once or not at all; the second parenthesis – that 2 appears once or never, etc. For partitions into odd parts, we have:

$$N(z) = \left(1 + z^2 + z^3 + \ldots\right)\left(1 + z^3 + z^6 + \ldots\right)\left(1 + z^5 + z^{10} + \ldots\right)\ldots \quad .$$

The interpretation of respective terms in parentheses is similar as in the case of the function $R(z)$. It is sufficient now to replace a term $1 + z^k$ in the generating function $R(z)$ by

$$\frac{1 - z^{2k}}{1 - z^k}.$$

To obtain the equality $R(z) = N(z)$.

2. We now focus on restricted partitions, i.e., a natural number *n* is to be written as a sum of predetermined parts. They are a useful tool to analyse monetary systems (Smoluk 1998). One of the first European monetary systems was created by an Athenian lawmaker, Solon (it was also a system of weights and measures), who derived it from the Babylonian number system based on sixty (threescore). Since about 594 BC, a Greek, or Attic talent was equal to 60 minas or 3600 shekels or 6000 drachmae (its weight was about 25.9 kilograms). After the coinage reform, one drachma coin contained 4.37 grams of silver. Another circulating coin was the obol whose name stemmed from a long thin metal rod used as a spit. Six obols were worth one drachma. Next, there was a stater (literally “weight”) minted of gold, electrum or silver. A gold and electrum stater was worth twenty drachmae, while a silver one – four drachmae (Hammond 1994). Let assume an obol as a basic coin. Then the Greek monetary system is described by four generating functions:

$$A_i(z) = 1 + z^i + z^{2i} + z^{3i} + \ldots,$$

where $i \in \{1, 6, 24, 120\}$ (Wilkowski 1998). Let $A(z) = 1 + a_1 z + a_2 z^2 + \ldots$ denote a product of the four above power series. In order to determine in how many ways one could pay, say, 3600 obols (that was a price of a good slave around 400 BC), (Mizerski 1996), we have to obtain the coefficient a_{3600} of series $A(z)$. It is the number of all feasible partitions of the number 3600 in the Greek monetary system. We may use available numeric software packages such as Matlab or Mathematica. A derivative of the 3600th order of the function $A(z)$ at zero is to be calculated and divided by the factorial 3600!

It is well-known that good money is strong and stable. However, all currencies tend to lose their real purchasing power, e.g., between the 9th and the 20th centuries the British pound's loss of purchasing power was as 1:3360, or in other words, the British pound in 1988 was worth merely 0.03 per cent of its original value (Żabiński 1989). Perhaps, the euro will be such a currency, since the GDP of the European Union's countries in 2009 exceeded that of the US. The euro may also become, in addition to the US dollar, a global reserve currency. Likewise, Athenian drachmae, deniers of Charlemagne, Florentine florins, Dutch guilders, or finally, Reichsthaler played such a role in the past.

The euro, just like the majority of other monetary systems globally, is based on a decimal system. One euro is divided into 100 cents. We have got coins in 1, 2, 5, 10, 20 and 50 cent denominations. The coins are issued in 1 and 2 euro denominations as well (since 2005, the European Commission has been thinking about the introduction of banknotes in these denominations, but no final decision was reached so far), while banknotes are issued in €5, €10, €20, €50, €100, €200 and €500.

Problem (Wilkowski 2000). In how many ways one can change a 10 euro note using only 1, 5, 10 and 50 cent coins?

Solution I (theoretical). The generating function for this situation has the following form:

$$A(z) = \left(1 + z + z^2 + \ldots\right)\left(1 + z^5 + z^{10} + \ldots\right)\left(1 + z^{10} + z^{20} + \ldots\right)\left(1 + z^{50} + z^{100} + \ldots\right).$$

In other words:

$$A(z) = \frac{1}{1-z} \cdot \frac{1}{1-z^5} \cdot \frac{1}{1-z^{10}} \cdot \frac{1}{1-z^{50}}.$$

We have to determine the coefficient a_{1000} of the term z^{1000} of the function $A(z)$. It is seen that

$$A(z) = \left(1 + z + z^2 + z^3 + z^4\right) B\left(z^5\right),$$

where

$$B(z) = \frac{1}{1-z} \cdot \frac{1}{1-z} \cdot \frac{1}{1-z^2} \cdot \frac{1}{1-z^{10}} .$$

Since each term in the denominator of the function $B(z)$ is a divisor of $1 - z^{10}$, hence we get:

$$B(z) = \frac{W(z)}{(1-z^{10})^4} .$$

A polynomial $W(z)$ has the following form:

$$W(z) = (1 + z + \ldots + z^9)^2(1 + z^2 + \ldots + z^8) =$$

$$= 1 + 2z + 4z^2 + 6z^3 + 9z^4 + 12z^5 + 16z^6 + 20z^7 + 25z^8 + 30z^9 +$$

$$+ 33z^{10} + 36z^{11} + 37z^{12} + 38z^{13} + 37z^{14} + \ldots + 2z^{25} + z^{26},$$

$$(w_i = w_{26-i},\ \ i = 0, 1, \ldots , 26).$$

It is known (Graham, Knuth, Patashnik 1998) that

$$\frac{1}{(1-z^{10})^4} = \sum_{n=0}^{\infty} \binom{n+3}{3} z^{10n},$$

therefore

$$B(z) = \left(\sum_{n=0}^{\infty} \binom{n+3}{3} z^{10n} \right) W(z).$$

We can now calculate the coefficient b_n appearing with z^n in the series $B(z)$. With $n = 10k + i$ and $0 \le i < 10$, we have:

$$b_{10k+i} = w_i \binom{k+3}{3} + w_{i+10} \binom{k+2}{3} + w_{i+20} \binom{k+1}{3},$$

where w_i is the coefficient of the polynomial

$$W(z) = w_0 + w_1 z + \ldots + w_{26} z^{26}.$$

Because $a_{50k} = b_{10k}$ therefore

$$a_{1000} = \binom{23}{3} + 33\binom{22}{3} + 16\binom{21}{3} = 73871.$$

With 1, 5, 10 and 50 cent coins available, we can change a 10 euro banknote in 73871 ways.

Solution II (numerical). Using Matlab or Mathematica, we obtain the solution:

$$a_{1000} = \frac{A(0)^{(1000)}}{1000!} = 73871.$$

To conclude this section, let us note that in view of the ongoing development of cashless payments, increasing usage of credit, charge and debit cards and the like, perhaps the banknotes of high denominations will become redundant. However, it is less likely that all coins and notes will completely disappear along with current monetary systems that we study by means of generating functions.

3. When solving some probabilistic problems, we face the failure of intuition. Such problems include among others: the Chebyshev problem, the Bertrand paradox (Wilkowski 2007), the Penney's game or the Monty Hall paradox (Dniestrzański, Wilkowski 2008). This section deals with this type of a problem, i.e., the birthday paradox. In this problem we ask a question: what is the smallest number n of people that at least one pair of them will have the same birthday with the probability greater than 0.5? Students of many US universities were asked this question, with an average answer 385 (Nikodem 2010), thus proving that their intuition failed (it is interesting what the outcome would be among Polish students).

Let us therefore solve the problem contained in this question. We assume that birthdays are uniformly distributed on the set {1, …, 365} (we disregard leap years, twins and seasonal variation of births). These factors do not significantly affect the solution. Elementary events are now k-term sequences whose elements are from the set {1, …, 365}. There are 365^k such sequences. Let p_k denote the probability of the complement of the event under study, i.e., the event that among the *k* persons each one has a different birthday. Then, it holds:

$$p_k = \frac{365 \cdot 364 \cdot \ldots \cdot (365 - k + 1)}{365^k} = \left(1 - \frac{1}{365}\right) \cdot \ldots \cdot \left(1 - \frac{k-1}{365}\right),$$

for $2 \le k \le 365$ and

$$p_k = 0$$

for $k > 365$.

Using the inequality $1 + x \le e^x$ that is true for any real number x, we obtain the following approximation:

$$p_k \le e^{-\frac{1}{365}} \cdot \ldots \cdot e^{-\frac{k-1}{365}} = e^{-\frac{k(k-1)}{730}}.$$

The smallest natural k with $p_k \le ½$ is equal 23. Using a software package such as Matlab or Mathematica yields the following approximation:

$$p_{23} \approx 0{,}492703, \qquad p_{22} \approx 0{,}524305.$$

Hence, if there are at least 23 persons in a group, then a probability that at least two of them have the same birthday is greater than ½.

Let us now consider a more difficult problem (assuming, as before, the uniformity, and disregarding leap years, twins and triplets). What is the smallest number of persons so that the probability that at least three of them share a birthday is greater than 0.5? The probability of the complementary event equals:

$$Pr \text{ (no more than two out of } k \text{ persons share a birthday)} = \frac{f_k}{365^k},$$

where

$$f_k = \sum_{l \ge 0} \binom{365}{k-l} \binom{k-l}{l} \frac{k!}{2^l} \text{ (Adamaszek 2010).}$$

We seek the smallest natural number k that satisfies the inequality:

$$1 - \frac{f_k}{365^k} > \frac{1}{2}.$$

Using the software package Matlab or Mathematica, we obtain:

$$1 - \frac{f_{88}}{365^{88}} = 0{,}511\ldots\ .$$

We conclude that at least 88 persons are needed for the probability that at least three of them have the same birthday to exceed ½. During class sessions, one can solve similar problems requesting the four, five and more persons sharing the same birthday.

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